

Complex Numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} \quad \text{where } \underline{i^2 = -1}.$$

Note \mathbb{C} has two operations:

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + \underline{(b_1 + b_2) i}$$

$$\begin{aligned} (a_1 + b_1 i) \cdot (a_2 + b_2 i) &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i^2 \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i \end{aligned}$$

Observation: when $b_1 = 0$, $a_1 (a_2 + b_2 i) = (a_1 a_2) + (a_1 b_2) i$

The Complex numbers form a (real) vector space!

Even better: Use complex numbers instead of real numbers when defining vector spaces...

This yields Complex vector spaces!

Ex: $\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : a, b, c \in \mathbb{C} \right\} = \mathbb{C}^3$

NB: Everything we've done so far can be extended to complex vector spaces as well 😊.

Point: Don't be afraid of complex numbers...

Last Time: The eigenvalues of a matrix M are the roots of the characteristic polynomial $\underline{p_M(\lambda)}$.
$$p_M(\lambda) = \det(M - \lambda I)$$

Ex: Compute E-values of $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Sol:
$$P_M(\lambda) = \det \left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$
$$= \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}$$
$$= (1-\lambda)^2 - (1 \cdot -1) = (1-\lambda)^2 + 1$$

$$\begin{aligned} \therefore P_M(\lambda) = 0 &\iff (1-\lambda)^2 + 1 = 0 \\ &\iff (1-\lambda)^2 = -1 \\ &\iff 1-\lambda = \pm i \\ &\iff \lambda = 1 \pm i \end{aligned}$$

$\therefore M$ has complex eigenvalues!



Q: Given an E-value, what are its eigenvectors?

Ex: Consider $M = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$.

$$\begin{aligned} P_M(\lambda) &= \det \begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix} \\ &= (3-\lambda)(2-\lambda) - 2 \\ &= 6 - 5\lambda + \lambda^2 - 2 \\ &= \lambda^2 - 5\lambda + 4 \\ &= (\lambda-4)(\lambda-1) \end{aligned}$$

$$\therefore P_M(\lambda) = 0 \iff \lambda = 4 \text{ or } \lambda = 1.$$

Because E-vectors must satisfy

$$Mv = \lambda v \quad \text{i.e.} \quad (M - \lambda I)v = 0$$

$$\text{i.e.} \quad v \in \text{null}(M - \lambda I),$$

we can find E-vectors by computing $\text{null}(M - \lambda I)$!

$$\text{For } \lambda = 4: M - \lambda I = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \overset{x}{-1} & \overset{y}{1} & | & 0 \\ 2 & -2 & | & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \overset{x}{-1} & \overset{y}{1} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \quad \leftarrow \text{solving } (M - 4I) \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

when $-x + y = 0$ we have solution!

Point: $\begin{bmatrix} x \\ x \end{bmatrix}$ should be an eigenvector for $\lambda = 4$

$$\text{Check: } \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 4x \\ 4x \end{bmatrix} = 4 \begin{bmatrix} x \\ x \end{bmatrix} \quad \checkmark$$

$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis of eigenspace of $\lambda = 4$.

(Recall: Eigenspace associated to λ is $V_\lambda := \{v \in V : Mv = \lambda v\}$)

For $\lambda = 1$: Compute $\text{null}(M - 1I)$

$$M - I = \begin{bmatrix} 3-1 & 1 \\ 2 & 2-1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \overset{x}{2} & \overset{y}{1} & | & 0 \\ 2 & 1 & | & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \rightsquigarrow \begin{cases} 2x + y = 0 \\ 0 = 0 \end{cases} \rightsquigarrow y = -2x$$

Thus $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ forms a basis for E-space V_1 .

Check: $M \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 2-4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \checkmark$

Hence, we have $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ a basis of eigenvectors of M for $\mathbb{R}^2 \dots$

On a whim: Let's compute $\text{Rep}_{B,B}(L_M)$.

where $\text{Rep}_{E_2, E_2}(L_M) = M$:

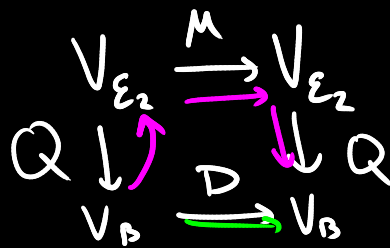
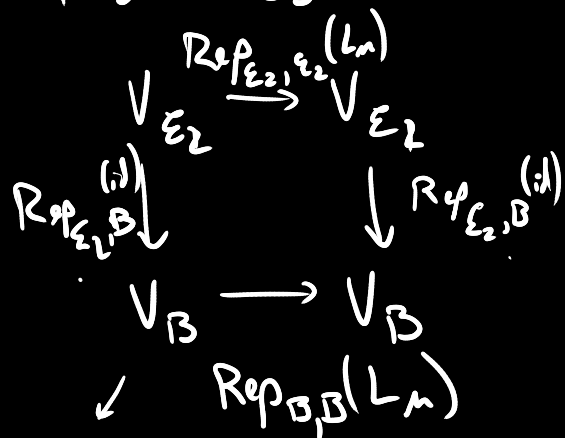
* $\text{Rep}_{E_2, B}(\text{id}) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \text{Rep}_{B, E_2}(\text{id})$.

Compute:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 1 \end{array} \right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \end{array} \right]$$

$$\therefore \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}}_Q^{-1} = \frac{1}{3} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}}_{Q^{-1}} \checkmark$$



$$\begin{aligned} \boxed{D} &= Q^{-1} M Q \\ &= \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 8 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \end{aligned}$$

$$= \frac{1}{3} \begin{bmatrix} 12 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

is a diagonal matrix w/ E-values on diag...

Q: When is M "diagonalizable"?

Defn: Two $n \times n$ matrices A, B are similar when there is an invertible matrix P such that $A = P^{-1} B P$. Matrix M is diagonalizable when there is a diagonal matrix D to which M is similar.

Ex: We just showed $M = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ is similar to $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = D$, so M is diag'ble.

In general, it turns out M ^{$n \times n$} is diagonalizable
 * if and only if \mathbb{R}^n has a basis of E-vectors of M .

IDEA: $M = P^{-1}DP$

means M and D rep. same transf. $\mathbb{R}^n \rightarrow \mathbb{R}^n$ with different bases...

Indeed, $P = \text{Rep}_{B, B'}(\text{id}) \dots$

The E-vectors of M and D are the same...

In particular, for $v \in B'$ $\text{Rep}_{B'}(v) = e_i$

$$D \text{Rep}_{B'}(v) = D e_i = d_{ii} e_i$$

\uparrow i th entry on diag of D !

Thus v is an eigenvector for the transformation

D represents! Thus B' is a basis

for \mathbb{R}^n consisting entirely of E-vectors of L .

Computationally: we can check if M is diag'ble by checking if E-vectors of M contain a basis for \mathbb{R}^n ...

↳ ① Compute $P_M(\lambda)$.

② Find E-values (via $P_M(\lambda) = 0$)

③ Compute E-vectors For Each λ .

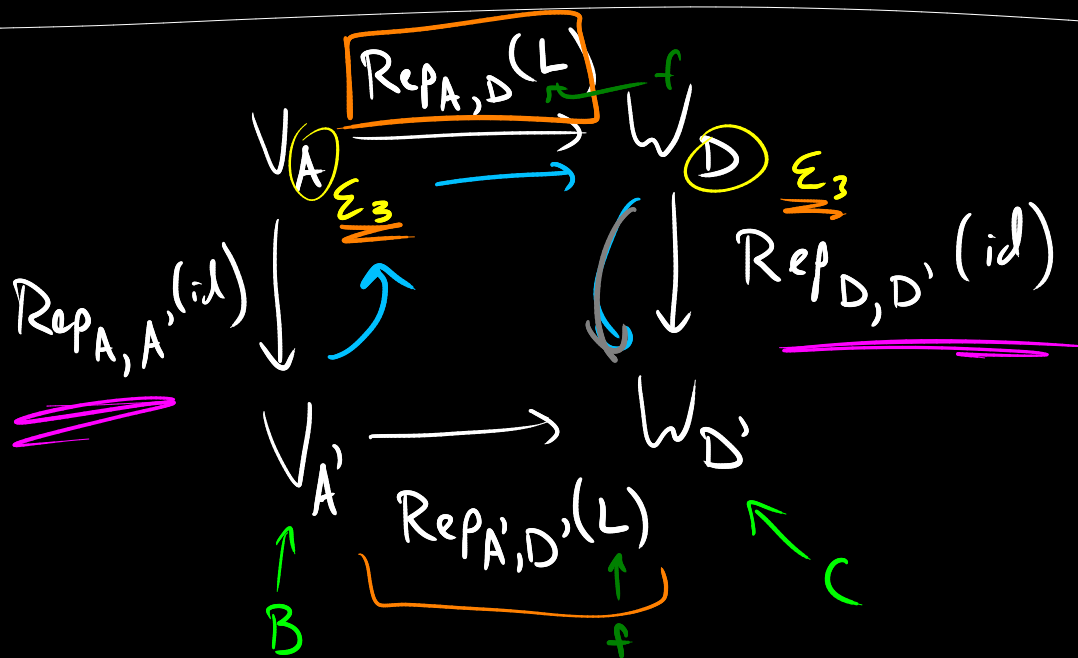
(via solving $(M - \lambda I)\vec{x} = \vec{0}$ and computing a basis of the corresp. spe).

* ④ Check that these bases together form a basis for \mathbb{R}^n ...

Lem: If M is a matrix w/ distinct E-values λ_1 and λ_2 , then the E-spaces V_{λ_1} and V_{λ_2} have only the 0-vector in common.
 i.e. any bases for V_{λ_1} and V_{λ_2} are lin. indep. of one another...

\therefore Part ④ becomes:

④' There are n lin. indep E-vectors of M .



$$\text{Rep}_{B,C}(f) = \text{Rep}_{\epsilon_3,C}(\text{id}) \cdot \text{Rep}_{\epsilon_3,\epsilon_3}(f) \cdot \text{Rep}_{B,\epsilon_3}(\text{id})$$

$$[f]_B^C = \text{Rep}_{B,C}(f).$$